

Global Solutions of the Navier-Stokes Equations for Isentropic Flow with Large External Potential Force

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To my family and my wife Candy

Abstract. We prove the global-in-time existence of weak solutions to the Navier-Stokes equations of compressible isentropic flow in three space dimensions with adiabatic exponent $\gamma \geq 1$. Initial data and solutions are small in L^2 around a non-constant steady state with densities being positive and essentially bounded. No smallness assumption is imposed on the external forces when $\gamma = 1$. A great deal of information about partial regularity and large-time behavior is obtained.

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1. Introduction

We prove the global existence of weak solutions to the Navier-Stokes equations of compressible flow in three space dimensions:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u^j)_t + \operatorname{div}(\rho u^j u) + (P_\gamma)_{x_j} = \mu \Delta u + (\xi - \mu) (\operatorname{div} u)_{x_j} + f^j. \end{cases} \quad (1.1)$$

Here ρ and $u = (u^1, u^2, u^3)$ are the unknown functions of $x \in \mathbb{R}^3$ and $t \geq 0$, $P_\gamma = P_\gamma(\rho)$ is the pressure, f is the external force, μ and $\xi - \mu$ are viscosity constants.

The system (1.1) is solved subject to initial conditions

$$(\rho(\cdot, 0), u(\cdot, 0)) = (\rho_0, u_0), \quad (1.2)$$

where ρ_0 is bounded above and below away from zero, $u_0 \in L^p$ for some $p > 6$, and modulo constants, $(\rho_0 - \tilde{\rho}_\gamma, u_0)$ is small in $L^2(\mathbb{R}^3)$ for some specific non-constant function $\tilde{\rho}_\gamma$ which will be defined later. The solvability to various Cauchy problems for the Navier-Stokes equations has been discussed by many other mathematicians for decades. Matsumura-Nishida [9] proved the global existence of H^3 solutions around a constant state when the initial data is taken to be small in H^3 , and later Danchin [1] generalized their results by replacing H^3 with certain Besov spaces of functions. On the other hand, Lions [8] and Feireisl [2]-[3] proved global existence of weak solutions to (1.1)-(1.2) with finite-energy initial data and nonnegative initial density. In between those two type of solutions as mentioned, Hoff [4]-[6] developed a new theory of *intermediate regularity* class solutions which may exhibit discontinuities in density and velocity gradient across hypersurfaces in \mathbb{R}^3 . In the presence of large external force, our work generalizes earlier results of Matsumura-Yamagata [10] in two ways: the restriction on the L^∞ norm of $\rho - \tilde{\rho}_\gamma$ has been eliminated and initial velocity is not necessary in H^1 .

We introduce two variables associated with the system (1.1) which are important to our analysis. The first one is the usual vorticity matrix $\omega = \omega^{j,k} = u_{x_k}^j - u_{x_j}^k$, while the other one is the effective

viscous flux F given by

$$F = \xi \operatorname{div} u - P_\gamma(\rho) + P_\gamma(\tilde{\rho}_\gamma). \quad (1.3)$$

By adding and subtracting terms, we can rewrite the momentum equation as in (1.1) in terms of F and ω :

$$\rho \dot{u}^j = F_{x_j} + \mu \omega_{x_k}^{j,k} + \rho f^j - P_\gamma(\tilde{\rho}_\gamma)_{x_j}. \quad (1.4)$$

The decomposition (1.4) also implies that

$$\Delta F = \operatorname{div}(\rho \dot{u} + \rho f - \nabla P_\gamma(\tilde{\rho}_\gamma)). \quad (1.5)$$

We refer to Hoff [4] for a more detailed discussion of F .

We now give a precise formulation of our results. First concerning the pressure P_γ we assume that

(1.6) there is $L > 0$ such that $P_\gamma(\rho) = L\rho^\gamma$ for $\gamma \geq 1$.

We also fix a positive reference density ρ_∞ and choose bounding densities $0 < \underline{\rho} < \bar{\rho}$ satisfying

$$\underline{\rho} < \rho_\infty < \bar{\rho} \quad (1.7)$$

and we define a positive number δ such that

$$\delta = \frac{1}{2} \min\{|\rho_\infty - \underline{\rho}|, |\rho_\infty - \bar{\rho}|\}. \quad (1.8)$$

Concerning the external force f , we assume that

(1.9) $f = -\rho \nabla \psi$, where $\psi \in H^4(\mathbb{R}^3)$;

(1.10) there is a constant $\mathcal{C} > 0$ such that $|\psi(x)| + |\nabla \psi(x)| \leq \mathcal{C}$ and $|D_x^2 \psi(x)| \leq |\nabla \psi(x)|$ for $x \in \mathbb{R}^3$;

(1.11) $\lim_{|x| \rightarrow \infty} \psi(x) = 0$.

Concerning the diffusion coefficients μ and ξ , we assume that

$$0 < \xi < \left(\frac{3}{2} + \frac{\sqrt{21}}{6}\right)\mu. \quad (1.12)$$

It follows that

$$\frac{1}{4}\mu(p-2) - \frac{[\frac{1}{4}(\xi - \mu)(p-2)]^2}{\frac{1}{3}\mu + (\xi - \mu)} > 0 \quad (1.13)$$

for $p = 6$ and consequently for some $p > 6$, which we now fix.

We define $\tilde{\rho}_\gamma$ as mentioned at the beginning of this section. Given a positive constant density ρ_∞ , we say $(\tilde{\rho}_\gamma, 0)$ is a *steady state solution* to (1.1) if $\tilde{\rho}_\gamma \in C^2(\mathbb{R}^3)$ and the following holds

$$\begin{cases} \nabla P_\gamma(\tilde{\rho}(x)) = -\tilde{\rho}_\gamma(x) \nabla \psi(x), \\ \lim_{|x| \rightarrow \infty} \tilde{\rho}(x) = \rho_\infty. \end{cases} \quad (1.14)$$

By direct computation, ρ_γ has the following explicit form

$$(1.15) \quad \rho_\gamma(x) = \begin{cases} \rho_\infty \exp\left[-\frac{1}{L}\psi(x)\right] & \text{if } \gamma = 1, \\ \left[\rho_\infty^{\gamma-1} - \frac{\gamma-1}{L\gamma}\psi(x)\right]^{\frac{1}{\gamma-1}} & \text{if } \gamma > 1. \end{cases}$$

Concerning the initial data (ρ_0, u_0) we assume that there is a positive number N , which may be arbitrarily large such that

$$\|u_0\|_{L^p} \leq N, \quad (1.16)$$

and there is positive number δ with $d < \delta$ such that

$$\underline{\rho} + d < \operatorname{ess\,inf} \rho_0 \leq \operatorname{ess\,sup} \rho_0 < \bar{\rho} - d, \quad (1.17)$$

We also write

$$C_0 = \|\rho_0 - \tilde{\rho}_\gamma\|_{L^2} + \|u_0\|_{L^2}. \quad (1.18)$$

where $\tilde{\rho}_\gamma$ is defined as in (1.15).

Weak solutions are defined in the usual way; we say that (ρ, u) is a weak solution of (1.1)-(1.2) provided that $(\rho - \tilde{\rho}_\gamma, \rho u) \in C([0, \infty); H^{-1}(\mathbb{R}^3))$ with $(\rho, u)|_{t=0} = (\rho_0, u_0)$, $\nabla u \in L^2(\mathbb{R}^3 \times (0, \infty))$ for $t > 0$, and the following identities hold for times $t_2 \geq t_1 \geq 0$ and C^1 test functions φ having uniformly bounded support in x for $t \in [t_1, t_2]$:

$$\int_{\mathbb{R}^3} \rho(x, \cdot) \varphi(x, \cdot) dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho \varphi_t + \rho u \cdot \nabla \varphi) dx dt, \quad (1.19)$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} (\rho u^j)(x, \cdot) \varphi(x, \cdot) dx \Big|_{t_1}^{t_2} &= \int_{t_1}^{t_2} \int_{\mathbb{R}^3} [\rho u^j \varphi_t + \rho u^j u \cdot \nabla \varphi + P(\rho) \varphi_{x_j}] dx dt \\ &\quad - \int_{t_1}^{t_2} \int_{\mathbb{R}^3} [\mu \nabla u^j \cdot \nabla \varphi + (\mu - \xi)(\operatorname{div} u) \varphi_{x_j}] dx dt \\ &\quad + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \rho \varphi_{x_j} \psi dx dt. \end{aligned} \quad (1.20)$$

We use the usual notation for Hölder seminorms: for $v : \mathbb{R}^3 \rightarrow \mathbb{R}^m$ and $\alpha \in (0, 1]$,

$$\langle v \rangle^\alpha = \sup_{\substack{x_1, x_2 \in \mathbb{R}^3 \\ x_1 \neq x_2}} \frac{|v(x_2) - v(x_1)|}{|x_2 - x_1|^\alpha};$$

and for $v : Q \subseteq \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^m$ and $\alpha_1, \alpha_2 \in (0, 1]$,

$$\langle v \rangle_Q^{\alpha_1, \alpha_2} = \sup_{\substack{(x_1, t_1), (x_2, t_2) \in Q \\ (x_1, t_1) \neq (x_2, t_2)}} \frac{|v(x_2, t_2) - v(x_1, t_1)|}{|x_2 - x_1|^{\alpha_1} + |t_2 - t_1|^{\alpha_2}}.$$

We denote the material derivative of a given function v by $\dot{v} = v_t + \nabla v \cdot u$, and if X is a Banach space we will abbreviate X^3 by X when convenient. If $I \subset [0, \infty)$ is an interval, $C^1(I; X)$ will be the elements $v \in C(I; X)$ such that the distribution derivative $v_t \in \mathcal{D}'(\mathbb{R}^3 \times \operatorname{int} I)$ is realized as an element of $C(I; X)$. Finally, if $\Omega \subset \mathbb{R}^3$ is a Lebesgue measurable subset in \mathbb{R}^3 , $|\Omega|$ will be the corresponding volume of Ω .

The following is the main result of this paper, which is valid for the case when $\gamma = 1$:

Theorem 1.1 *Let $\gamma = 1$ and assume that the system parameters in (1.1) satisfy the conditions in (1.6) and (1.9)-(1.13). Let positive numbers $\mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}, \delta$ be given satisfying (1.7)-(1.8) and*

$$L \log \left(\frac{\rho_\infty}{\bar{\rho}} \right) < -\mathcal{C} < \mathcal{C} < L \log \left(\frac{\rho_\infty}{\underline{\rho}} \right) \quad (1.21)$$

Then given positive numbers N and $d < \delta$, there are positive constants a, C, θ depending on the parameters and assumptions in (1.6) and (1.9)-(1.13), on $\mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}, R$, on N and on a positive lower bound for d , such that if an initial data (ρ_0, u_0) is given satisfying (1.16)-(1.18) with

$$C_0 < a, \quad (1.22)$$

then there is a solution (ρ, u) to (1.1)-(1.2) in the sense of (1.19)-(1.20) on all of $\mathbb{R}^3 \times [0, \infty)$. The solution satisfies the following:

$$\rho - \tilde{\rho}_\gamma, \rho u \in C([0, \infty); H^{-1}(\mathbb{R}^3)), \quad (1.23)$$

$$\nabla u \in L^2(\mathbb{R}^3 \times (0, \infty)), \quad (1.24)$$

$$u(\cdot, t) \in H^1(\mathbb{R}^3), \quad t > 0, \quad (1.25)$$

$$\omega(\cdot, t), F(\cdot, t) \in H^1(\mathbb{R}^3), \quad t > 0, \quad (1.26)$$

$$\langle u \rangle_{\mathbb{R}^3 \times [\tau, \infty)}^{\frac{1}{2}, \frac{1}{4}} \leq C(\tau) C_0^\theta, \quad \tau > 0, \quad (1.27)$$

where $C(\tau)$ may depend additionally on a positive lower bound for τ ,

$$\underline{\rho} \leq \rho(x, t) \leq \bar{\rho} \text{ a.e. on } \mathbb{R}^3 \times [0, \infty), \quad (1.28)$$

and

$$\begin{aligned} & \sup_{t>0} \int_{\mathbb{R}^3} [(\rho - \tilde{\rho}_\gamma)^2 + |u|^2 + \sigma |\nabla u|^2 + \sigma^3 (F^2 + |\nabla \omega|^2)] dx \\ & + \int_0^\infty \int_{\mathbb{R}^3} [|\nabla u|^2 + \sigma(|\dot{u}|^2 + |\nabla \omega|^2) + \sigma^3 |\nabla \dot{u}|^2] dx ds \leq C C_0^\theta \end{aligned} \quad (1.29)$$

where $\sigma(t) = \min\{1, t\}$. Moreover, $(\rho, u) \rightarrow (\tilde{\rho}_\gamma, 0)$ as $t \rightarrow \infty$ in the sense that, for all $r_1 \in (2, \infty]$ and $r_2 \in (2, \infty)$,

$$\lim_{t \rightarrow \infty} \|\rho(\cdot, t) - \tilde{\rho}_\gamma(\cdot)\|_{L^{r_1}} + \|u(\cdot, t)\|_{L^{r_2}} = 0. \quad (1.30)$$

Using similar method, we also obtain parallel results for $\gamma > 1$, with an extra assumption on the support of the external potential force ψ :

Theorem 1.2 *Let $\gamma > 1$ and assume that the system parameters in (1.1) satisfy the conditions in (1.6) and (1.9)-(1.13). Let positive numbers $\mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}, \delta$ be given satisfying (1.7)-(1.8) and*

$$\frac{L\gamma}{\gamma-1} [\rho_\infty^{\gamma-1} - \bar{\rho}^{\gamma-1}] < -\mathcal{C} < \mathcal{C} < \frac{L\gamma}{\gamma-1} [\rho_\infty^{\gamma-1} - \underline{\rho}^{\gamma-1}] \quad (1.31)$$

Then given positive numbers N and $d < \delta$, there are positive constants a, C, θ depending on the parameters and assumptions in (1.6) and (1.9)-(1.13), on $\gamma, \mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}$, on N and on a positive lower bound for d , such that if an initial data (ρ_0, u_0) is given satisfying (1.16)-(1.18) and the small-energy assumption (1.22), and if

$$|\text{supp}(\psi)| < a, \quad (1.32)$$

then there is a solution (ρ, u) to (1.1)-(1.2) in the sense of (1.19)-(1.20) on all of $\mathbb{R}^3 \times [0, \infty)$. The solution satisfies (1.23)-(1.27) with bounds (1.28)-(1.29), and $(\rho, u) \rightarrow (\tilde{\rho}_\gamma, 0)$ as $t \rightarrow \infty$ in the sense of (1.30) for all $r_1 \in (2, \infty]$ and $r_2 \in (2, \infty)$.

We point out that for the case when $\gamma = 1$, system (1.1) becomes simpler than those for $\gamma > 1$.

For example, in deriving *a priori* bounds for the term $\int_0^t \int_{\mathbb{R}^3} \sigma |\dot{u}|^2 dx ds$, we first rewrite the momentum equation in (1.1) as follows

$$\rho \dot{u} + \tilde{P}_\gamma \nabla (P_\gamma \tilde{P}_\gamma^{-1} - 1) + \rho \tilde{P}_\gamma^{-1} \nabla \tilde{P}_\gamma (\rho^{-1} P_\gamma - \tilde{\rho}_\gamma^{-1} \tilde{P}_\gamma) - \mu \Delta u - (\xi - \mu) \nabla (\text{div } u) = 0. \quad (1.33)$$

When $\gamma = 1$, the third of the above reads

$$\rho \tilde{P}_\gamma^{-1} \nabla \tilde{P}_\gamma (\rho^{-1} P_\gamma - \tilde{\rho}_\gamma^{-1} \tilde{P}_\gamma) = \rho \tilde{P}_\gamma^{-1} \nabla \tilde{P}_\gamma (L - L) = 0,$$

so if we multiply (1.33) with $\sigma \dot{u}$ and integrate, $\int_0^t \int_{\mathbb{R}^3} \sigma |\dot{u}|^2 dx ds$ can be bounded in terms of the L^2 norms of $(\rho - \tilde{\rho}, u)$ and the H^1 norms of u (modulo higher order terms in u).

On the other hand, when $\gamma > 1$, without any cancellation, we then have to estimate the term $\int_0^t \int_{\mathbb{R}^3} \sigma \dot{u} \rho \tilde{P}_\gamma^{-1} \nabla \tilde{P}_\gamma (\rho^{-1} P_\gamma - \tilde{\rho}_\gamma^{-1} \tilde{P}_\gamma) dx ds$ which makes the analysis much more intricate and forces extra assumptions imposed on the external force. We will explain more later in subsequent sections.

This paper is organized as follows. We begin the proofs of Theorem 1.1 and 1.2 in section 2 with a number of *a priori* bounds for local-in-time smooth solutions. Since many of these estimates are rather long and technical, we omit those which are identical to or nearly identical to arguments given elsewhere in the literature. In section 3 we derive the necessary bounds for density by applying

the estimates in Theorem 3.1 and 3.2 in a maximum principle argument along particle trajectories of the velocity, making important use of the monotonicity of P_γ as described in (1.8). The small-energy assumption (1.22) then enables us to close these arguments to show in Theorem 3.1 and 3.2 that both the pointwise bounds for density and the *a priori* energy bounds of Theorem 3.1 do hold as long as the smooth solution exists. Finally in section 4 we prove Theorem 1.1 and 1.2 by constructing weak solutions as limits of smooth solutions corresponding to mollified initial data.

We make use of the following standard facts (see Ziemer [15] Theorem 2.1.4, Remark 2.4.3, and Theorem 2.4.4, also Ladyzhenskaya [7] section 1.4, for example). First, given $r \in [2, 6]$ there is a constant $C(r)$ such that for $w \in H^1(\mathbb{R}^3)$,

$$\|w\|_{L^r(\mathbb{R}^3)} \leq C(r) \left(\|w\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r} \|\nabla w\|_{L^2(\mathbb{R}^3)}^{(3r-6)/2r} \right). \quad (1.34)$$

Next, for any $r \in (1, \infty)$ there is a constant $C(r)$ such that for $w \in W^{1,r}(\mathbb{R}^3)$,

$$\|w\|_{L^\infty(\mathbb{R}^3)} \leq C(r) \|w\|_{W^{1,r}(\mathbb{R}^3)} \quad (1.35)$$

and

$$\langle w \rangle_{\mathbb{R}^3}^\alpha \leq C(r) \|\nabla w\|_{L^r(\mathbb{R}^3)}, \quad (1.36)$$

where $\alpha = 1 - 3/r$. If Γ is the fundamental solution of the Laplace operator on \mathbb{R}^3 , then there is a constant C such that for any $f \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$,

$$\|\Gamma_{x_j} * f\|_{L^\infty(\mathbb{R}^3)} \leq C (\|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^4(\mathbb{R}^3)}). \quad (1.37)$$

Finally, there is a constant M such that for any $v \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \frac{|v(x)|^2}{(1+|x|)^2} dx \leq M \int_{\mathbb{R}^3} |\nabla v|^2 dx. \quad (1.38)$$

2. Energy Estimates

In this section we derive *a priori* bounds for smooth, local-in-time solutions of (1.1)-(1.2) whose densities are strictly positive and bounded. Specifically, we fix a smooth solution $(\rho - \tilde{\rho}_\gamma, u)$ of (1.1)-(1.2) on $\mathbb{R}^3 \times [0, T]$ for some time $T > 0$ with initial data $(\rho_0 - \tilde{\rho}_\gamma, u_0)$. These bounds will depend only on the quantities $C_0, \mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}, N, d$ and will be independent of the initial regularity and the time of existence.

We define a functional $A(t)$ for a given such solution by

$$\begin{aligned} A(t) = & \sup_{0 < s \leq t} \int_{\mathbb{R}^3} [\sigma |\nabla u|^2 + \sigma^3 (|\dot{u}|^2 + |\nabla \omega|^2)] dx \\ & + \int_0^t \int_{\mathbb{R}^3} [\sigma (|\dot{u}|^2 + |\nabla \omega|^2) + \sigma^3 |\nabla \dot{u}|^2] dx ds, \end{aligned} \quad (2.1)$$

where $\sigma(t) \equiv \min\{1, t\}$, and we obtain the following *a priori* bound for $A(t)$ under the assumptions that the initial energy C_0 in (1.18) is small and that the density remains bounded above and below away from zero when $\gamma = 1$:

Theorem 2.1 *Let $\gamma = 1$. Assume that the system parameters in (1.1) satisfy the conditions in (1.6) and (1.9)-(1.13). Let positive numbers $\mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}, \delta$ be given satisfying (1.7)-(1.8) and (1.21). Then given positive numbers N and $d < \delta$, there are positive constants a, M, θ depending on the parameters and assumptions in (1.6) and (1.9)-(1.13), on $\mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}$, on N and on a positive lower bound for d , such that if $(\rho - \tilde{\rho}_\gamma, u)$ is a solution of (1.1)-(1.2) on $\mathbb{R}^3 \times [0, T]$ with initial data $(\rho_0 - \tilde{\rho}_\gamma, u_0) \in H^3(\mathbb{R}^3)$ satisfying (1.16)-(1.18) with $C_0 < a$, and if*

$$\underline{\rho} \leq \rho(x, t) \leq \bar{\rho} \text{ on } \mathbb{R}^3 \times [0, T],$$

then

$$A(T) \leq MC_0^\theta.$$

With an extra assumption on $\text{supp}(\psi)$, we can also obtain the following estimate for $\gamma > 1$:

Theorem 2.2 *Let $\gamma > 1$. Assume that the system parameters in (1.1) satisfy the conditions in (1.6) and (1.9)-(1.13). Let positive numbers $\mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}, \delta$ be given satisfying (1.7)-(1.8) and (1.31). Then given positive numbers N and $d < \delta$, there are positive constants a, M, θ depending on the parameters and assumptions in (1.6) and (1.9)-(1.13), on $\gamma, \mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}$, on N and on a positive lower bound for d such that if $|\text{supp}(\psi)| < a$ and if $(\rho - \tilde{\rho}_\gamma, u)$ is a solution of (1.1)-(1.2) on $\mathbb{R}^3 \times [0, T]$ with initial data $(\rho_0 - \tilde{\rho}_\gamma, u_0) \in H^3(\mathbb{R}^3)$ satisfying (1.16)-(1.18) with $C_0 < a$ and*

$$\underline{\rho} \leq \rho(x, t) \leq \bar{\rho} \text{ on } \mathbb{R}^3 \times [0, T],$$

then

$$A(T) \leq MC_0^\theta.$$

The proof will be given in a sequence of lemmas in which we estimate a number of auxiliary functionals. To describe these we first recall the definition (1.13) of p , which is an open condition, and which therefore allows us to choose $q \in (6, p)$ which also satisfies (1.13). Then for a given $(\rho - \tilde{\rho}_\gamma, u)$, we define

$$H(t) = \int_0^t \int_{\mathbb{R}^3} \left[\sigma^{3/2} |\nabla u|^3 + \sigma^3 |\nabla u|^4 \right] dx ds + \left| \sum_{1 \leq k_i, j_m \leq 3} \int_0^t \int_{\mathbb{R}^3} \sigma u_{x_{k_1}}^{j_1} u_{x_{k_2}}^{j_2} u_{x_{k_3}}^{j_3} dx ds \right|, \quad (2.2)$$

$$D(t) = \int_0^t \int_{\mathbb{R}^3} |\rho - \tilde{\rho}_\gamma|^2 |\nabla P(\tilde{\rho})|^2 dx ds. \quad (2.3)$$

Also, we can readily see that assumptions (1.21) and (1.31) imply $\tilde{\rho}_\gamma$ is well-defined and

$$\underline{\rho} < \tilde{\rho}_\gamma < \bar{\rho}, \quad (2.4)$$

which will be crucial to the later analysis.

For simplicity, we write $\sigma = \sigma(t) = \min\{1, t\}$, $P_\gamma = P_\gamma(\rho)$ and $\tilde{P}_\gamma = P_\gamma(\tilde{\rho}_\gamma)$ without further referring.

We begin with the following L^2 energy estimate, which is valid for $\gamma \geq 1$:

Lemma 2.3 *Assume that the system parameters in (1.1) satisfy the conditions in (1.6) and (1.9)-(1.13). Then if $(\rho - \tilde{\rho}_\gamma, u)$ is a solution of (1.1)-(1.2) on $\mathbb{R}^3 \times [0, T]$ with initial data $(\rho_0 - \tilde{\rho}_\gamma, u_0) \in H^3(\mathbb{R}^3)$ satisfying (1.16)-(1.18), and if $\rho, \tilde{\rho} \in [\underline{\rho}, \bar{\rho}]$, then for $\gamma \geq 1$,*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} (|\rho - \tilde{\rho}_\gamma|^2 + \rho |u|^2) dx + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 dx dt \leq MC_0. \quad (2.5)$$

Proof. We use $\nabla \psi = -\tilde{\rho}_\gamma^{-1} \nabla \tilde{P}_\gamma$ on the momentum equation to get

$$\rho \dot{u} + \rho(\rho^{-1} \nabla P_\gamma - \tilde{\rho}_\gamma^{-1} \nabla \tilde{P}_\gamma) - \mu \Delta u - (\xi - \mu) \nabla(\text{div } u) = 0. \quad (2.6)$$

Multiply the above by u and integrate to obtain that for $0 \leq t \leq T$,

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{1}{2} \rho |u|^2 dx \Big|_0^t + \int_0^t \int_{\mathbb{R}^3} \rho u (\rho^{-1} \nabla P_\gamma - \tilde{\rho}_\gamma^{-1} \nabla \tilde{P}_\gamma) dx ds \\ + \int_0^t \int_{\mathbb{R}^3} [\mu |\nabla u|^2 + (\xi - \mu)(\operatorname{div} u)^2] dx ds = 0, \end{aligned} \quad (2.7)$$

where the divergence of a matrix is taken row-wise. Next we define

$$G(\rho) = \int_{\tilde{\rho}_\gamma}^\rho \int_{\tilde{\rho}_\gamma}^r s^{-1} P'_\gamma(s) ds dr,$$

then using the mass equation, the second term on the left side of (2.7) can be written as follows

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \rho u (\rho^{-1} \nabla P_\gamma - \tilde{\rho}_\gamma^{-1} \nabla \tilde{P}_\gamma) dx ds \\ = \int_0^t \int_{\mathbb{R}^3} \rho u \cdot \nabla \left(\int_{\tilde{\rho}_\gamma}^\rho r^{-1} P'_\gamma(r) dr \right) dx ds \\ = \int_0^t \int_{\mathbb{R}^3} \rho_t \left(\int_{\tilde{\rho}_\gamma}^\rho r^{-1} P'_\gamma(r) dr \right) dx ds = \int_0^t \int_{\mathbb{R}^3} G(\rho)_s dx ds = \int_{\mathbb{R}^3} G(\rho) dx \Big|_0^t. \end{aligned}$$

Putting the above into (2.7), the result follows. \square

Next we derive preliminary L^2 bounds for ∇u and \dot{u} :

Lemma 2.4 *Assume that the hypotheses and notations of Lemma 2.3 are in force. Then for $t \in (0, T]$ and $\gamma = 1$,*

$$\sup_{0 < s \leq t} \sigma \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma \rho |\dot{u}|^2 dx ds \leq M [C_0 + H], \quad (2.8)$$

and

$$\sup_{0 < s \leq t} \sigma^3 \int_{\mathbb{R}^3} |\dot{u}|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla \dot{u}|^2 dx ds \leq M [C_0 + H]; \quad (2.9)$$

while for $\gamma > 1$,

$$\sup_{0 < s \leq t} \sigma \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma \rho |\dot{u}|^2 dx ds \leq M [C_0 + H + D], \quad (2.10)$$

and

$$\sup_{0 < s \leq t} \sigma^3 \int_{\mathbb{R}^3} |\dot{u}|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla \dot{u}|^2 dx ds \leq M [C_0 + H]. \quad (2.11)$$

Proof. The proofs are nearly the same as those of (2.9) and (2.12) in [4], except that $\tilde{\rho}_\gamma$ is not necessarily a constant. We prove (2.8) and (2.10) as an example. First, (2.6) can be rewritten as follows

$$\rho \dot{u} + \tilde{P}_\gamma \nabla (P_\gamma \tilde{P}_\gamma^{-1} - 1) + \rho \tilde{P}_\gamma^{-1} \nabla \tilde{P}_\gamma (\rho^{-1} P_\gamma - \tilde{\rho}_\gamma^{-1} \tilde{P}_\gamma) - \mu \Delta u - (\xi - \mu) \nabla (\operatorname{div} u) = 0. \quad (2.12)$$

We multiply (2.12) by $\sigma \dot{u}$ and integrate to obtain

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \sigma \rho |\dot{u}|^2 dx ds - \mu \int_0^t \int_{\mathbb{R}^3} \sigma \dot{u} dx ds - (\xi - \mu) \int_0^t \int_{\mathbb{R}^3} \sigma \dot{u} \nabla(\operatorname{div} u) dx ds \\ & - \int_0^t \int_{\mathbb{R}^3} \sigma \dot{u} \tilde{P}_\gamma \nabla(P_\gamma \tilde{P}_\gamma^{-1} - 1) dx ds \\ & - \int_0^t \int_{\mathbb{R}^3} \sigma \dot{u} \rho \tilde{P}_\gamma^{-1} \nabla \tilde{P}_\gamma (\rho^{-1} P_\gamma - \tilde{\rho}_\gamma^{-1} \tilde{P}_\gamma) dx ds = 0. \end{aligned} \quad (2.13)$$

The second and the third term on the left side of (2.13) can be bounded above by

$-\frac{\mu}{2} \sigma \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{\xi - \mu}{2} \sigma \int_{\mathbb{R}^3} (\operatorname{div} u)^2 dx + H$. The forth term on the left side of (2.13) can be estimated as follows

$$\begin{aligned} \left| - \int_0^t \int_{\mathbb{R}^3} \sigma \dot{u} \tilde{P}_\gamma \nabla(P_\gamma \tilde{P}_\gamma^{-1} - 1) dx ds \right| & \leq M \left[\sigma \int_{\mathbb{R}^3} |\nabla u| |\rho - \tilde{\rho}_\gamma| dx + \int_0^{1 \wedge t} \int_{\mathbb{R}^3} |\nabla u| |\rho - \tilde{\rho}_\gamma| dx ds \right. \\ & \quad \left. + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx \right], \end{aligned}$$

where the right side is bounded by $M[C_0 + A]$ by Lemma 2.3. Finally for the fifth term, we notice that for $\gamma = 1$,

$$\rho^{-1} P_\gamma - \tilde{\rho}_\gamma^{-1} \tilde{P}_\gamma = L - L = 0,$$

so that the term vanishes for the case when $\gamma = 1$. On the other hand, for $\gamma > 1$, using the definition (2.3) of D , we have

$$\begin{aligned} & - \int_0^t \int_{\mathbb{R}^3} \sigma \dot{u} \rho \tilde{P}_\gamma^{-1} \nabla \tilde{P}_\gamma (\rho^{-1} P_\gamma - \tilde{\rho}_\gamma^{-1} \tilde{P}_\gamma) dx ds \\ & \leq M \int_0^t \int_{\mathbb{R}^3} \sigma |\dot{u}| |\phi(\gamma)| |\rho - \tilde{\rho}_\gamma| |\nabla \tilde{\rho}_\gamma| dx ds \leq D^{\frac{1}{2}} \left[\int_0^t \int_{\mathbb{R}^3} \sigma |\dot{u}|^2 dx ds \right]^{\frac{1}{2}}. \end{aligned}$$

Therefore (2.8) and (2.10) follow. \square

The following auxiliary estimates will be applied to bound the the functional H :

Lemma 2.5 *Assume that the hypotheses and notations of Lemma 2.3 are in force. Then for $0 < t \leq 1 \wedge T$ and $\gamma \geq 1$,*

$$\begin{aligned} & \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |u|^q dx + \int_0^t \int_{\mathbb{R}^3} [|u|^{q-2} |\nabla u|^2 + |u|^{q-4} |\nabla(|u|^2)|^2] dx ds \\ & \leq M \left[C_0^{\frac{p-q}{p-2}} N^{\frac{q-2}{p-2}} + C_0 \right]. \end{aligned} \quad (2.14)$$

Proof. We multiply (2.12) by $u|u|^{q-2}$ and integrate to obtain that

$$\begin{aligned} & \int_{\mathbb{R}^3} |u|^q dx + \int_0^t \int_{\mathbb{R}^3} |u|^{q-2} |\nabla u|^2 dx ds + \int_0^t \int_{\mathbb{R}^3} |u|^{q-4} |\nabla(|u|^2)|^2 dx ds \\ & \leq M \left[\int_0^t \int_{\mathbb{R}^3} (P_\gamma - \tilde{P}_\gamma) \operatorname{div}(|u|^{q-2} u) dx ds + \int_0^t \int_{\mathbb{R}^3} |u|^{q-1} |\nabla \tilde{\rho}_\gamma| |P_\gamma - \tilde{P}_\gamma| dx ds \right. \\ & \quad \left. - \int_0^t \int_{\mathbb{R}^3} \rho \tilde{P}_\gamma^{-1} (\rho^{-1} P_\gamma - \tilde{\rho}_\gamma^{-1} \tilde{P}_\gamma) |u|^{q-2} u dx ds + \int_{\mathbb{R}^3} |u_0|^q dx \right]. \end{aligned} \quad (2.15)$$

The last term on the right side of (2.15) is bounded by $C_0^{\frac{p-q}{p-2}} N^{\frac{q-2}{p-2}}$, and the first and second term can be bounded in terms of $\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\rho - \tilde{\rho}_\gamma|^2 dx$. For the remaining term we have

$$\begin{aligned} & - \int_0^t \int_{\mathbb{R}^3} \rho \tilde{P}_\gamma^{-1} (\rho^{-1} P_\gamma - \tilde{\rho}_\gamma^{-1} \tilde{P}_\gamma) |u|^{q-2} u dx ds \\ & \leq \int_0^t \int_{\mathbb{R}^3} |\nabla \tilde{\rho}_\gamma| \phi(\gamma) |\rho - \tilde{\rho}_\gamma| |u|^{q-1} dx ds \\ & \leq M \left[\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\rho - \tilde{\rho}_\gamma|^2 \right]^{\frac{1}{q}} \left[\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |u|^q dx \right]^{\frac{q-1}{q}}. \end{aligned}$$

Using the above on (2.15) and absorbing terms, the result follows. \square

We obtain bounds for u, ω and F in $W^{1,r}$, these being required for the derivation of estimates for the auxiliary functionals H and D :

Lemma 2.6 *Let $\gamma \geq 1$ and assume that the hypotheses and notations of Lemma 2.3 are in force. Then for $r \in (1, \infty)$ and $t \in (0, T]$,*

$$\|\nabla u(\cdot, t)\|_{L^r} \leq M \left[\|F(\cdot, t)\|_{L^r} + \|\omega(\cdot, t)\|_{L^r} + \|(P_\gamma - \tilde{P}_\gamma)(\cdot, t)\|_{L^r} \right]. \quad (2.16)$$

The constant M in (2.16) may depend additionally on r .

Proof. We have from the definition (1.3) of F that

$$\xi \Delta u^j = F_{x_j} + \xi \omega_{x_k}^{j,k} + (P_\gamma - \tilde{P}_\gamma)_{x_j}. \quad (2.17)$$

Differentiating and taking the Fourier transform we then obtain

$$\xi \widehat{u}_{x_l}^j(y, t) = \frac{y_j y_l}{|y|^2} \widehat{F}(y, t) + \xi \frac{y_k y_l}{|y|^2} \widehat{\omega^{j,k}}(y, t) + \frac{y_k y_l}{|y|^2} \widehat{(P_\gamma - \tilde{P}_\gamma)}(y, t).$$

The result (2.16) then follows immediately from the Marcinkiewicz multiplier theorem (Stein [12], pg. 96). \square

Lemma 2.7 *Assume that the hypotheses and notations of Lemma 2.3 are in force. Then for $\gamma > 1$,*

$$\begin{aligned} & \|\nabla F(\cdot, t)\|_{L^2} + \|\nabla \omega(\cdot, t)\|_{L^2} \\ & \leq M \left[\|\dot{u}(\cdot, t)\|_{L^2} + \|\nabla u(\cdot, t)\|_{L^2} + \|\nabla \tilde{P} \|\rho - \tilde{\rho}_\gamma(\cdot, t)\|_{L^2} \right]; \end{aligned} \quad (2.18)$$

while for $\gamma = 1$,

$$\|\nabla \left(\frac{F}{\rho} \right) (\cdot, t)\|_{L^2} + \|\nabla \left(\frac{\omega}{\tilde{\rho}} \right) (\cdot, t)\|_{L^2} \leq M \left[\|\dot{u}(\cdot, t)\|_{L^2} + \|\nabla u(\cdot, t)\|_{L^2} \right]. \quad (2.19)$$

Proof. For (2.18), using (1.5) we have

$$\Delta F = (\rho \dot{u}^j)_{x_j} + [(\tilde{P}_\gamma)_{x_j} \tilde{\rho}^{-1} (\tilde{\rho}_\gamma - \rho)]_{x_j}. \quad (2.20)$$

Similar to (2.17), we differentiate and take Fourier transform on (2.20) and apply the multiplier theorem, we get the bounds for ∇F and similarly for $\text{div } \omega$. To show (2.19), using (1.3) and the

definition (1.4) of F , for $\gamma = 1$,

$$\begin{aligned}
\rho \ddot{u}^j &= \left(\frac{F}{L\tilde{\rho}} L\tilde{\rho} \right)_{x_j} + \mu \omega_k + \rho \tilde{\rho}^{-1} L\tilde{\rho}_{x_j} - L\tilde{\rho}_{x_j} \\
&= \tilde{\rho} \left(\frac{F}{\tilde{\rho}} \right)_{x_j} + L\tilde{\rho}_{x_j} \left[\frac{\xi}{L\tilde{\rho}} \operatorname{div} u - \rho \tilde{\rho}^{-1} + 1 \right] + \mu \left(\frac{\omega}{L\tilde{\rho}} L\tilde{\rho} \right)_{x_k} + L\tilde{\rho}_{x_j} (\rho \tilde{\rho}^{-1} - 1) \\
&= \tilde{\rho} \left(\frac{F}{\tilde{\rho}} \right)_{x_j} + \frac{\mu}{L} \left(\frac{\omega}{\tilde{\rho}} \right)_{x_k} + \frac{\tilde{\rho}_{x_j} \xi}{\tilde{\rho}} \operatorname{div} u + \frac{\tilde{\rho}_{x_j} \mu}{\tilde{\rho}} \omega,
\end{aligned} \tag{2.21}$$

and so,

$$\|\nabla \left(\frac{F}{\tilde{\rho}} \right) (\cdot, t)\|_{L^2} \leq M \left[\|\dot{u}(\cdot, t)\|_{L^2} + \|\nabla u(\cdot, t)\|_{L^2} + \|\nabla \left(\frac{\omega}{\tilde{\rho}} \right) (\cdot, t)\|_{L^2} \right]. \tag{2.22}$$

Using similar method, we obtain

$$\|\nabla \left(\frac{\omega}{\tilde{\rho}} \right) (\cdot, t)\|_{L^2} \leq M [\|\dot{u}(\cdot, t)\|_{L^2} + \|\nabla u(\cdot, t)\|_{L^2}]. \tag{2.23}$$

Therefore (2.19) follows from (2.22) and (2.23). \square

Next we derive a bound for the functional D defined above in (2.3):

Lemma 2.8 *Assume that the hypotheses and notations of Lemma 2.3 are in force. Then for $\gamma = 1$ and $t \in (0, T]$,*

$$D(t) \leq M[C_0 + A(t)]. \tag{2.24}$$

For $\gamma > 1$, (2.24) also holds when $|\operatorname{supp}(\psi)|$ is sufficiently small.

Proof. In view of Lemma 2.3, it suffices to consider $t > 1$. We first consider the case when $\gamma = 1$. Using the mass equation,

$$\xi \frac{D}{Dt} (\rho - \tilde{\rho}_\gamma) + \rho (P_\gamma - \tilde{P}_\gamma) = -\rho \tilde{\rho} \frac{F}{\tilde{\rho}} - \xi u \cdot \nabla \tilde{\rho}_\gamma, \tag{2.25}$$

so that we multiply the above by $(\rho - \tilde{\rho}_\gamma) \operatorname{sgn}(\rho - \tilde{\rho}_\gamma) |\nabla \tilde{P}|^2$ and integrate to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\rho - \tilde{\rho}_\gamma|^2 |\nabla \tilde{P}|^2 dx + M^{-1} \int_1^t \int_{\mathbb{R}^3} |\rho - \tilde{\rho}_\gamma|^2 |\nabla \tilde{P}|^2 dx ds \\
&\leq M \left[\int_1^t \int_{\mathbb{R}^3} (|\frac{F}{\tilde{\rho}}|^2 + |u|^2) (1 + |x|)^{-2} dx ds \right] \\
&\leq M \left[\int_1^t \int_{\mathbb{R}^3} (|\nabla \left(\frac{F}{\tilde{\rho}} \right)|^2 + |\nabla u|^2) dx ds \right],
\end{aligned} \tag{2.26}$$

where the last line follows by (1.38). Using (2.5) and (2.19), the right side of (2.26) is bounded by $M[C_0 + A]$, and hence (2.24) holds for $\gamma = 1$.

For $\gamma > 1$, using (1.38), (2.18) and (2.26),

$$\begin{aligned}
&\int_1^t \int_{\mathbb{R}^3} |\rho - \tilde{\rho}_\gamma|^2 |\nabla \tilde{P}|^2 dx ds \leq M \left[C_0 + \int_1^t \int_{\mathbb{R}^3} (|F|^2 + |u|^2) |\nabla \tilde{P}|^2 dx ds \right] \\
&\leq M \left[C_0 + \int_1^t \int_{\mathbb{R}^3} |\nabla u|^2 dx ds + \int_1^t \int_{\mathbb{R}^3} |F|^2 |\nabla \tilde{P}|^2 dx ds \right] \\
&\leq M \left[C_0 + \int_1^t \int_{\operatorname{supp}(\psi)} |F|^2 |\nabla \tilde{P}|^2 dx ds \right],
\end{aligned} \tag{2.27}$$

where the last inequality follows from the fact that $|\nabla \tilde{P}| = \tilde{\rho}|\nabla \psi|$. Assume that $|\text{supp}(\psi)| < \infty$, the last term on the right side of (2.27) can be estimated as follows

$$\begin{aligned} \int_1^t \int_{\text{supp}(\psi)} |F|^2 |\nabla \tilde{P}|^2 dx ds &\leq |\text{supp}(\psi)|^{\frac{1}{3}} \int_1^t \left[\int_{\text{supp}(\psi)} |F|^3 |\nabla \tilde{P}|^3 dx \right]^{\frac{2}{3}} ds \\ &\leq |\text{supp}(\psi)|^{\frac{1}{3}} \int_1^t \left[\int_{\mathbb{R}^3} |F|^2 |\nabla \tilde{P}|^2 dx \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^3} |\nabla F|^2 |\nabla \tilde{P}|^2 dx + \int_{\mathbb{R}^3} |F|^2 |D_x^2 \tilde{P}|^2 dx \right]^{\frac{1}{2}} ds \\ &\leq |\text{supp}(\psi)|^{\frac{1}{3}} \left[\int_1^t \int_{\mathbb{R}^3} |F|^2 |\nabla \tilde{P}|^2 dx ds + M \int_1^t \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla u|^2 + |\nabla \tilde{P}|^2 |\rho - \tilde{\rho}|^2) dx ds \right] \\ &\leq |\text{supp}(\psi)|^{\frac{1}{3}} \left[\int_1^t \int_{\mathbb{R}^3} |F|^2 |\nabla \tilde{P}|^2 dx ds + M \int_1^t \int_{\mathbb{R}^3} |\nabla \tilde{P}|^2 |\rho - \tilde{\rho}|^2 dx ds + M(C_0 + A) \right], \end{aligned}$$

hence if $|\text{supp}(\psi)|$ is sufficiently small, we have

$$\int_1^t \int_{\mathbb{R}^3} |\rho - \tilde{\rho}_\gamma|^2 |\nabla \tilde{P}|^2 dx ds \leq M [C_0 + A]$$

and (2.24) follows. \square

We can now obtain the required estimates for the functionals H defined above in (2.2) :

Lemma 2.9 *Assume that the hypotheses and notations of Lemma 2.3 are in force. Then for $\gamma = 1$ and $t \in (0, T]$,*

$$H(t) \leq M [C_0 + C_0^2 + A(t)^2]. \quad (2.28)$$

For $\gamma > 1$, (2.28) also holds when $|\text{supp}(\psi)|$ is sufficiently small.

Proof. We first consider the case for $\gamma = 1$. In view of the definition (2.2) of H , we bound only the term

$$\int_0^t \int_{\mathbb{R}^3} [\sigma^{3/2} |\nabla u|^3 + \sigma^3 |\nabla u|^4] dx ds.$$

The remaining term $\sum_{1 \leq k_i, j_m \leq 3} \left| \int_0^t \int_{\mathbb{R}^3} \sigma u_{x_{k_1}}^{j_1} u_{x_{k_2}}^{j_2} u_{x_{k_3}}^{j_3} dx ds \right|$ is bounded exactly as in Hoff [4] pp. 29–32.

First we have from Lemma 2.6 that

$$\int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 dx ds \leq M \left[\int_0^t \int_{\mathbb{R}^3} \sigma^3 (|\rho - \tilde{\rho}_\gamma|^4 + |F|^4 + |\omega|^4) dx ds \right]. \quad (2.29)$$

Applying (1.34) we can bound the second term on the right by

$$\begin{aligned} \tilde{\rho}^4 \int_0^t \int_{\mathbb{R}^3} \sigma^3 \left| \frac{F}{\tilde{\rho}} \right|^4 dx ds &\leq \left(\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \sigma \left| \frac{F}{\tilde{\rho}} \right|^2 dx \right)^{\frac{1}{2}} \left(\sigma^3 \int_{\mathbb{R}^3} \left| \nabla \left(\frac{F}{\tilde{\rho}} \right) \right|^2 dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t \int_{\mathbb{R}^3} \sigma \left| \nabla \left(\frac{F}{\tilde{\rho}} \right) \right|^2 dx ds \right). \end{aligned}$$

Now from the definition of F and Lemma 2.3,

$$\left(\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \sigma \left| \frac{F}{\tilde{\rho}} \right|^2 dx \right)^{\frac{1}{2}} \leq M(C_0 + A)^{\frac{1}{2}}.$$

Also, from (2.19),

$$\int_0^t \int_{\mathbb{R}^3} \sigma \left| \nabla \left(\frac{F}{\tilde{\rho}} \right) \right|^2 dx ds \leq M \left[\int_0^t \int_{\mathbb{R}^3} \sigma (|\dot{u}|^2 + |\nabla u|^2) dx ds \right] \leq M [A + C_0]$$

and

$$\sigma^3 \int_{\mathbb{R}^3} |\nabla F|^2 dx \leq M \left[\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \sigma^3 (|\dot{u}|^2 + |\nabla u|^2) dx \right] \leq MA.$$

Thus

$$\int_0^t \int_{\mathbb{R}^3} \sigma^3 |F|^4 \leq M [C_0 + A]^2. \quad (2.30)$$

Applying the results of Lemma 2.7 in a similar way, we obtain that

$$\begin{aligned} \bar{\rho}^4 \int_0^t \int_{\mathbb{R}^3} \sigma^3 \left| \frac{\omega}{\bar{\rho}} \right|^4 dx ds &\leq \left(\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \sigma \left| \frac{\omega}{\bar{\rho}} \right|^2 dx \right)^{\frac{1}{2}} \left(\sigma^3 \int_{\mathbb{R}^3} \left| \nabla \left(\frac{\omega}{\bar{\rho}} \right) \right|^2 dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t \int_{\mathbb{R}^3} \sigma \left| \nabla \left(\frac{\omega}{\bar{\rho}} \right) \right|^2 dx ds \right) \\ &\leq M [C_0 + A]^2. \end{aligned} \quad (2.31)$$

For the first term on the right of (2.29), multiply (2.32) by $\sigma^3 |\rho - \tilde{\rho}_\gamma|^3 \text{sgn}(\rho - \tilde{\rho}_\gamma)$, integrate and use (2.30),

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \sigma^3 |\rho - \tilde{\rho}_\gamma|^4 dx ds &\leq M \left[C_0 + \int_0^t \int_{\mathbb{R}^3} \sigma^3 (|F|^4 + |u|^4) dx ds \right] \\ &\leq M [C_0 + (C_0 + A)^2]. \end{aligned} \quad (2.32)$$

Hence we obtain a bound for $\int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 dx ds$ from (2.30), (2.31) and (2.32). Bounds for the term $\int_0^t \int_{\mathbb{R}^3} \sigma^{3/2} |\nabla u|^3 dx ds$ are obtained in a similar way, thereby proving (2.28) for $\gamma = 1$.

For $\gamma > 1$, using (2.18),

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \sigma^3 |F|^4 dx ds &\leq \left(\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \sigma |F|^2 dx \right)^{\frac{1}{2}} \left[\int_0^t \left(\int_{\mathbb{R}^3} \sigma |\nabla F|^2 dx \right)^{\frac{3}{2}} ds \right] \\ &\leq (C_0 + A)^{\frac{1}{2}} \left[\int_0^t \left(\int_{\mathbb{R}^3} \sigma (|\dot{u}|^2 + |\nabla u|^2 + |\rho - \tilde{\rho}|^2 |\nabla \tilde{P}|^2) dx \right)^{\frac{3}{2}} ds \right] \\ &\leq M (C_0 + A)^{\frac{1}{2}} (A^{\frac{3}{2}} + A^{\frac{1}{2}} C_0 + C_0^{\frac{1}{2}} D) \\ &\leq M (C_0 + A + D)^2, \end{aligned}$$

and similarly,

$$\int_0^t \int_{\mathbb{R}^3} \sigma^3 |\rho - \tilde{\rho}_\gamma|^4 dx ds + \int_0^t \int_{\mathbb{R}^3} \sigma^3 |\omega|^4 dx ds \leq M (C_0 + A + D)^2.$$

So when $|\text{supp}(\psi)|$ is sufficiently small, we can apply (2.24) to conclude that

$$\int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 dx ds \leq M [C_0 + (C_0 + A)^2]$$

which proves (2.28) for $\gamma > 1$. □

Proof of Theorem 2.1 and 2.2: Theorem 2.1-2.2 now follows immediately from the bounds (2.5), (2.8), (2.9), (2.14), (2.24) and (2.28), and the fact that the functional A is continuous in time. □

3. Pointwise bounds for the density

In this section we derive pointwise bounds for the density ρ for $\gamma \geq 1$, bounds which are independent both of time and of initial smoothness. This will then close the estimates of Theorem 2.1-2.2 to give an uncontingent estimate for the functional A defined in (2.1). The result is as follows:

Theorem 3.1 *Let $\gamma = 1$. Assume that the system parameters in (1.1) satisfy the conditions in (1.6) and (1.9)-(1.13). Let positive numbers $\mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}, \delta$ be given satisfying (1.7)-(1.8) and (1.21). Then given positive numbers N and $d < \delta$, there are positive constants a, M, θ depending on the parameters and assumptions in (1.6) and (1.9)-(1.13), on $\mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}$, on N and on a positive lower bound for d , such that if $(\rho - \tilde{\rho}_\gamma, u)$ is a solution of (1.1)-(1.2) on $\mathbb{R}^3 \times [0, T]$ with initial data $(\rho_0 - \tilde{\rho}_\gamma, u_0) \in H^3(\mathbb{R}^3)$ satisfying (1.16)-(1.18) with $C_0 < a$, then in fact*

$$\underline{\rho} \leq \rho(x, t) \leq \bar{\rho} \text{ on } \mathbb{R}^3 \times [0, T],$$

and

$$A(T) \leq MC_0^\theta.$$

Theorem 3.2 *Let $\gamma > 1$. Assume that the system parameters in (1.1) satisfy the conditions in (1.6) and (1.9)-(1.13). Let positive numbers $\mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}, \delta$ be given satisfying (1.7)-(1.8) and (1.31). Then given positive numbers N and $d < \delta$, there are positive constants a, M, θ depending on the parameters and assumptions in (1.6) and (1.9)-(1.13), on $\gamma, \mathcal{C}, L, \rho_\infty, \bar{\rho}, \underline{\rho}$, on N and on a positive lower bound for d such that if $|\text{supp}(\psi)| < a$ and if $(\rho - \tilde{\rho}_\gamma, u)$ is a solution of (1.1)-(1.2) on $\mathbb{R}^3 \times [0, T]$ with initial data $(\rho_0 - \tilde{\rho}_\gamma, u_0) \in H^3(\mathbb{R}^3)$ satisfying (1.16)-(1.18) with $C_0 < a$, then in fact*

$$\underline{\rho} \leq \rho(x, t) \leq \bar{\rho} \text{ on } \mathbb{R}^3 \times [0, T],$$

and

$$A(T) \leq MC_0^\theta.$$

Proof. We prove Theorem 3.1 as an example, the proof of Theorem 3.2 is just similar. Most of the details are reminiscent of those in Hoff-Suen [13] section 3. First, we choose positive numbers b and b' satisfying

$$\underline{\rho} < b < \underline{\rho} + d < \bar{\rho} - d < b' < \bar{\rho}.$$

Recall that ρ_0 takes values in $[\underline{\rho} + d, \bar{\rho} - d]$, so that $\rho \in [\underline{\rho}, \bar{\rho}]$ on $\mathbb{R}^3 \times [0, \tau]$ for some positive τ . It then follows from Theorem 2.1 that $A(\tau) \leq MC_0^\theta$, where M is now fixed. We shall show that if C_0 is further restricted, then in fact $b < \rho < b'$ on $\mathbb{R}^3 \times [0, \tau]$, and so by a simple open-closed argument that $b < \rho < b'$ on all of $\mathbb{R}^3 \times [0, T]$, we have $A(T) \leq MC_0^\theta$ as well. We shall prove the required upper bound, the proof of the lower bound being similar.

Fix $y \in \mathbb{R}^3$ and define the corresponding particle path $x(t)$ by

$$\begin{cases} \dot{x}(t) = u(x(t), t) \\ x(0) = y. \end{cases}$$

Suppose that there is a time $t_1 \leq \tau$ such that $\rho(x(t_1), t_1) = b'$. We may take t_1 minimal and then choose $t_0 < t_1$ maximal such that $\rho(x(t_0), t_0) = \bar{\rho} - d$. Thus $\rho(x(t), t) \in [(\bar{\rho} - d), b']$ for $t \in [t_0, t_1]$. We consider two cases:

Case 1: $t_0 < t_1 \leq T \wedge 1$.

We have from the definition (1.3) of F and the mass equation that

$$\mu \frac{d}{dt} [\log \rho(x(t), t)] + P(\rho(x(t), t)) - P_\gamma(\tilde{\rho}_\gamma(x(t))) = -F(x(t), t).$$

Integrating from t_0 to t_1 and abbreviating $\rho(x(t), t)$ by $\rho(t)$, etc., we then obtain

$$\mu[\log(b') - \log(\bar{\rho} - d)] + \int_{t_0}^{t_1} [P_\gamma(s) - \tilde{P}_\gamma(s)] ds = - \int_{t_0}^{t_1} F(s) ds. \quad (3.1)$$

We shall show that

$$- \int_{t_0}^{t_1} F(s) ds \leq \tilde{M} C_0^\theta \quad (3.2)$$

for a constant \tilde{M} which depends on the same quantities as the M from Theorem 2.1 (which has been fixed). If so, then from (3.1),

$$\mu[\log(b') - \log(\bar{\rho} - d)] \leq \tilde{M} C_0^\theta, \quad (3.3)$$

where the last inequality holds because $P_\gamma(s) - \tilde{P}_\gamma(s)$ is nonnegative on $[t_0, t_1]$. But (3.3) cannot hold if C_0 is small depending on \tilde{M} , b' , and $\bar{\rho} - d$. Stipulating this smallness condition, we therefore conclude that there is no time t_1 such that $\rho(t_1) = \rho(x(t_1), t_1) = b'$. Since $y \in \mathbb{R}^3$ was arbitrary, it follows that $\rho < b'$ on $\mathbb{R}^3 \times [0, \tau]$, as claimed. The proof that $b < \rho$ is similar.

To prove (3.2) we let Γ be the fundamental solution of the Laplace operator in \mathbb{R}^3 and apply (1.5) to write

$$\begin{aligned} - \int_{t_0}^{t_1} F(s) ds &= - \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j}(x(s) - y) \rho \dot{u}^j(y, s) dy ds \\ &\quad - \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j}(x(s) - y) \left[(\tilde{P}_\gamma)_{x_j} \tilde{\rho}_\gamma^{-1} (\tilde{\rho}_\gamma - \rho) \right] dy ds. \end{aligned} \quad (3.4)$$

The first integral on the right here is bounded exactly as in Lemma 4.2 of Hoff [4]:

$$\begin{aligned} &\int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j}(x(t) - y) \rho \dot{u}^j(y, t) dy ds \\ &\leq \|\Gamma_{x_j} * (\rho u^j)(\cdot, t_1)\|_{L^\infty} + \|\Gamma_{x_j} * (\rho u^j)(\cdot, t_0)\|_{L^\infty} \\ &\quad + \left| \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j x_k}(x(s) - y) [u^k((x(s), s) - u^k(y, s))] (\rho u^j)(y, s) dy ds \right| \\ &\leq \tilde{M} C_0^\theta, \end{aligned}$$

and the second integral on the right side of (3.4) can be bounded as follows

$$\begin{aligned} &- \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j}(x(s) - y) [(\tilde{P}_\gamma)_{x_j} \tilde{\rho}_\gamma^{-1} (\tilde{\rho}_\gamma - \rho)] dy ds \\ &\leq \tilde{M} \int_0^1 \int_{\mathbb{R}^3} |y|^{-2} |\rho - \tilde{\rho}_\gamma|^2 dy ds \\ &\leq \tilde{M} \int_0^1 \left[\int_{|y| \leq 1} |y|^{-\frac{5}{2}} dy \right]^{\frac{4}{5}} \left[\int_{|y| \leq 1} |\rho - \tilde{\rho}_\gamma|^5 dy \right]^{\frac{1}{5}} ds \\ &\quad + \tilde{M} \int_0^1 \left[\int_{|y| > 1} |y|^{-4} dy \right]^{\frac{1}{2}} \left[\int_{|y| > 1} |\rho - \tilde{\rho}_\gamma|^2 dy \right]^{\frac{1}{2}} ds \\ &\leq \tilde{M} C_0^\theta, \end{aligned}$$

where the last inequality follows from Theorem 2.1. Thus (3.2) is proved.

Case 2: $1 \leq t_0 < t_1$.

Again by the mass equation and the definition (1.3) of F ,

$$\begin{aligned} \frac{d}{dt}(\rho(t) - \tilde{\rho}_\gamma(t)) + \mu^{-1}\rho(t)(P_\gamma(t) - \tilde{P}_\gamma(t)) \\ = -\mu^{-1}\rho(t)F(t) - \mu^{-1}u(t) \cdot \nabla \tilde{\rho}_\gamma(t). \end{aligned}$$

Multiplying by $(\rho(t) - \tilde{\rho}_\gamma(t))^3$ we get

$$\begin{aligned} \frac{1}{4} \frac{d}{dt}(\rho(t) - \tilde{\rho}_\gamma(t))^4 + \mu^{-1}g(t)\rho(t)(\rho(t) - \tilde{\rho}_\gamma(t))^4 \\ = -\mu^{-1}\rho(t)(\rho(t) - \tilde{\rho}_\gamma(t))^3 F(t) - \mu^{-1}u(t) \cdot \nabla \tilde{\rho}_\gamma(t)(\rho(t) - \tilde{\rho}_\gamma(t))^3, \end{aligned} \quad (3.5)$$

where $g(t) = (P_\gamma(t) - \tilde{P}_\gamma(t))(\rho(t) - \tilde{\rho}_\gamma(t))^{-1} \geq 0$ on $[t_0, t_1]$. We integrate (3.5) to obtain

$$\begin{aligned} (\rho(t_1) - \tilde{\rho}_\gamma(t_1))^4 - (\rho(t_0) - \tilde{\rho}_\gamma(t_0))^4 \\ \leq \tilde{M} \int_{t_0}^{t_1} [\|F(\cdot, s)\|_\infty^4 + \|u(\cdot, t)\|_\infty^4] ds. \end{aligned} \quad (3.6)$$

We shall show that

$$\tilde{M} \int_{t_0}^{t_1} [\|F(\cdot, s)\|_\infty^4 + \|u(\cdot, t)\|_\infty^4] ds \leq \tilde{M}C_0^\theta, \quad (3.7)$$

so that from (3.6),

$$0 < (b' - \bar{\rho})^4 - (\bar{\rho} - d - \bar{\rho})^4 \leq \tilde{M}C_0^\theta.$$

This cannot hold if C_0 is sufficiently small, however, so that, as in Case 1, there is no time t_1 such that $\rho(t_1) = \rho(x(t_1), t_1) = b'$. Since $y \in \mathbb{R}^3$ was arbitrary, it follows that $\rho < b'$ on $\mathbb{R}^3 \times [0, \tau]$, as claimed.

To prove (3.7), it suffices to consider the term $\tilde{M} \int_{t_0}^{t_1} \|F(\cdot, s)\|_\infty^4 ds$. We apply (1.37) to get

$$\begin{aligned} \int_{t_0}^{t_1} \|F(\cdot, s)\|_\infty^2 ds \leq \int_{t_0}^{t_1} \left[\|\rho \dot{u}(\cdot, s)\|_{L^4}^2 + \|\nabla \tilde{P}_\gamma \tilde{\rho}_\gamma^{-1}(\tilde{\rho}_\gamma - \rho)(\cdot, s)\|_{L^4}^2 \right] ds \\ + \int_{t_0}^{t_1} \left[\|\rho \dot{u}(\cdot, s)\|_{L^2}^2 + \|\nabla \tilde{P}_\gamma \tilde{\rho}_\gamma^{-1}(\tilde{\rho}_\gamma - \rho)(\cdot, s)\|_{L^2}^2 \right] ds, \end{aligned} \quad (3.8)$$

and the right side of the above is readily seen to be bounded by $\tilde{M}C_0^\theta$. Thus (3.7) is proved. \square

4. Global Existence of Weak Solutions: Proof of Theorem 1.1-1.2

In this section we complete the proof of Theorem 1.1 and 1.2 by constructing weak solutions as limits of smooth solutions. Specifically we fix the constants a and M defined in Theorems 2.1-2.2 and 3.1-3.2, we let initial data (ρ_0, u_0) be given satisfying the hypotheses (1.16)-(1.18) and (1.22) of Theorem 1.1 and Theorem 1.2, and we take (ρ_0^η, u_0^η) to be smooth approximate initial data obtained by convolving (ρ_0, u_0) with a standard mollifying kernel of width $\eta > 0$. We apply the local existence results (see Nash [11] or Tani [14]) to show that there is a smooth local solution (ρ^η, u^η) of (1.1)-(1.2) with initial data (ρ_0^η, u_0^η) , defined up to a positive time T , which may depend on η . The *a priori* estimates of Theorem 3.1-3.2 then apply to show that

$$A(t) \leq MC_0^\theta \text{ and } \underline{\rho} \leq \rho^\eta(x, t) \leq \bar{\rho}, \quad (4.1)$$

where $A(t)$ is defined by (2.1) but with (ρ, u) replaced by (ρ^η, u^η) . By standard arguments together with the bounds (4.1), (ρ^η, u^η) exists and satisfies (1.1)-(1.2) for all time.

Those bounds in (4.1) will provide the compactness needed to extract the desired solution (ρ, u) in the limit as $\eta \rightarrow 0$. We begin by proving uniform Hölder continuity of the families $\{u^\eta\}$ away from $t = 0$:

Lemma 4.1 *Given $\tau > 0$ there is a constant $C = C(\tau)$ such that, for all $\eta > 0$,*

$$\langle u^\eta(\cdot, t) \rangle_{\mathbb{R}^3 \times [\tau, \infty)}^{\frac{1}{2}, \frac{1}{4}} \leq C(\tau) C_0^\theta. \quad (4.2)$$

Proof. The proof is exactly as in Hoff [4], pg. 33 and pg. 41–42. \square

Compactness of the approximate solutions (ρ^η, u^η) now follows:

Lemma 4.2 *There is a sequence $\eta_k \rightarrow 0$ and functions u and ρ such that as $k \rightarrow \infty$,*

$$u^{\eta_k} \rightarrow u \text{ uniformly on compact sets in } \mathbb{R}^3 \times (0, \infty); \quad (4.3)$$

$$\nabla u^{\eta_k}(\cdot, t), \nabla \omega^{\eta_k}(\cdot, t) \rightharpoonup \nabla u(\cdot, t), \nabla \omega(\cdot, t) \quad (4.4)$$

weakly in $L^2(\mathbb{R}^3)$ for all $t > 0$;

$$\sigma^{\frac{1}{2}} \dot{u}^{\eta_k}, \sigma^{\frac{3}{2}} \nabla \dot{u}^{\eta_k} \rightharpoonup \sigma^{\frac{1}{2}} \dot{u} \sigma^{\frac{3}{2}} \nabla \dot{u} \quad (4.5)$$

weakly in $L^2(\mathbb{R}^3 \times [0, \infty))$; and

$$\rho^{\eta_k}(\cdot, t) \rightarrow \rho(\cdot, t) \quad (4.6)$$

strongly in $L^2_{loc}(\mathbb{R}^3)$ for every $t \geq 0$.

Proof. The uniform convergence (4.3) follows from Lemma 4.1 via a diagonal process. The statements in (4.4) and (4.5) then follow for this same sequence from (4.1) and elementary considerations based on the equality of weak- L^2 derivatives and distribution derivatives. The convergence of approximate densities (4.6) for a further subsequence requires a more involved argument, given in Lions [8] pp. 21–23 and extended by Feireisl [2], pp. 63–64 and 118–127. \square

Proof of Theorem 1.1-1.2: We only prove Theorem 1.1 since Theorem 1.2 can be proved in a similar way. It is clear that the limiting functions (ρ, u) of Lemma 4.2 inherit the bounds in (1.24)–(1.29) from (4.1) and (4.2) (but notice that no statements are made in (1.28) concerning $\dot{u}(\cdot, t)$). It is also clear from the modes of convergence described in Lemma 4.2 that (ρ, u) satisfies the weak forms (1.19)–(1.20) of the differential equations in (1.1) as well as the initial condition (1.2). The continuity statement (1.23) then follows easily from these weak forms together with the bounds in (1.29).

It remains to show (1.30). By the same argument as in Hoff [4] pp. 44–47, we have for $r_2 \in (2, \infty)$,

$$\lim_{t \rightarrow \infty} \|\rho(\cdot, t) - \tilde{\rho}_\gamma(\cdot)\|_{L^\infty} + \|u(\cdot, t)\|_{L^{r_2}} = 0.$$

So using (1.29), for $r_1 \in (2, \infty)$,

$$\lim_{t \rightarrow \infty} \|\rho(\cdot, t) - \tilde{\rho}_\gamma(\cdot)\|_{L^{r_1}} \leq C C_0^\theta \lim_{t \rightarrow \infty} \|\rho(\cdot, t) - \tilde{\rho}_\gamma(\cdot)\|_{L^\infty} = 0.$$

This completes the proof of Theorem 1.1. \square

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